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Sheaf cohomology on sites and the Leray spectral sequence

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Contents

Introduction

Étale cohomology was introduced and developped by Alexander Grothendieck and his collaborators and used by Pierre Deligne to prove the Weil conjectures. It is a sheaf cohomology theory. Originally, sheaf cohomology was constructed as a cohomology theory on the category of sheaves on topological spaces. Grothendieck noticed that in order to define sheaves, one just needs a category having some appropriate properties, together with a notion of coverings for each of its objects, and sheaves are defined to be contravariant functors from this category satisfying a sheaf property with respect to these coverings. This led to the definition of sites. And étale cohomology is a version of sheaf cohomology on sites, notably on the étale site of a scheme. For a scheme, we can also study sheaf cohomology on its underlying topological space. It is also a version of sheaf cohomology on sites, because there is a site assigned to each topological space. So the theory of sheaf cohomology on sites can be viewed as a unifying theory of cohomologies. Not only it is unifying, it also provides a tool for seeing relationships between these theories, by defining an appropriate notion of continuous functions between sites.

In this thesis, I start from basics from abelian categories and homological algebra to construct the theory of sheaf cohomology on sites. In particular, I study explicitly how the theory works for the étale site of a scheme, without going in the depth of étale cohomology. In the last chapter of this thesis, I define spectral sequences and construct the Leray spectral sequence which is the main tool of comparison of cohomologies on different sites, provided a continuous function exists between them.

I would like to thank my thesis supervisor Dr. Gabriel Chênevert for his helpful suggestions, clarifications and numerous comments.

1 Preliminaries

In this section, we present general category theory language and important results that we need for the rest of our work. We do not provide proofs, unless the results are essential for the other sections. A reader familiar with category theory could skip this chapter. For more detailed definitions, examples, and proofs, one can check [8].

1.1 Categories and functors

A *category* $\mathcal{C}$ consists of a class of objects $\mathrm{Ob}(\mathcal{C})$ and for every two objects A, B of $\mathcal{C}$ of a set of morphisms $\mathrm{Hom}(A,B)$ such that:

- for every object A, an identity morphism $Id_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ is given;
- for any objects A, B, and C, a composition law is given as follows:

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

$$(f, g) \mapsto g \circ f$$

  which is associative and such that for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we have:

$$f \circ Id_A = Id_B \circ f = f.$$

For any morphism $f : A \to B$ in a category $\mathcal{C}$, A is called the *source* of f, and B is called the target of f.

Abelian groups, together with homomorphisms of abelian groups, form a category **Ab**.

The *opposite category* $\mathcal{C}^{op}$ of $\mathcal{C}$ is defined by:

- $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$;
- for every objects A, B in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Now, let $\mathcal{C}$ be a category, and let $f : A \to B$ be a morphism in $\mathcal{C}$. The morphism f is called *monomorphism* if for any two given morphisms $u, v : Z \to A$ in $\mathcal{C}$ satisfying $f \circ u = f \circ v$, we have $u = v$. It is called *epimorphism* if f is a monomorphism in the opposite category $\mathcal{C}^{op}$ of $\mathcal{C}$. It is called *isomorphism* if there exists a morphism $g : B \to A$ such that:

$$g \circ f = Id_A,$$

$$f \circ g = Id_B.$$

One can prove that if a morphism is an isomorphism, then it is both an epimorphism and a monomorphism, but the converse is generally not true.

Let $f : A \to B$ and $f' : A' \to B$ be monomorphisms in a category $\mathcal{C}$. We will say that f *dominates* f', denoted by $f \geq f'$ if there exists a morphism $u : A \to A'$ in $\mathcal{C}$ such that $f = f' \circ u$. f is said to be *equivalent* to f', denoted by $f \sim f'$, if $f' \geq f$ and $f \geq f'$; in this case u and u' (where $u' : A' \to A$ is the morphism such that $f' = f \circ u'$) are inverses of each other. The relation $\sim$ is an equivalence relation on the monomorphisms with target B. Its equivalence classes are called *subobjects* of B.

We proceed in a similar way to define quotients of an object in a category. Let $f : B \to A$ and $f' : B \to A'$ be epimorphisms in a category $\mathcal{C}$. Then, we say that $f \sim f'$ if there exist morphisms $u : A \to A'$ and $u' : A' \to A$ such that $f' = u \circ f$ and $f = u' \circ f'$. Again, $\sim$ is an equivalence relation on the epimorphisms with source B. Its equivalence classes are called *quotients* of B.

An object W of a category $\mathcal{C}$ is called:

- *initial* if $\mathrm{Hom}_{\mathcal{C}}(W, X)$ consists of only one element for each object X of $\mathcal{C}$;
- *terminal* if $\mathrm{Hom}_{\mathcal{C}}(X, W)$ consists of only one element for each object X of $\mathcal{C}$;
- a *zero object* if it is both intial and terminal.

If it exists, an initial (resp. terminal) object of a category $\mathcal{C}$ is unique up to a unique isomorphism.

A *covariant functor* F from a category $\mathcal{C}$ to a category $\mathcal{C}'$ consists of:

- a map from $\mathrm{Ob}(\mathcal{C})$ to $\mathrm{Ob}(\mathcal{C}')$, that we denote also by F;
- for any two objects A, B of $\mathcal{C}$, a map from $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to $\mathrm{Hom}_{\mathcal{C}'}(F(A), F(B))$ that preserves the identity morphisms and composition, and that we denote also by F.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{C}'$.

A *natural transformation* of covariant functors (resp. of contravariant functors) $\alpha : F \to F'$ on a category $\mathcal{C}$ with target a category $\mathcal{C}'$ consists of a family of morphisms $F(A) \to F'(A)$ for every object A of $\mathcal{C}$ such that the diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha(A)} & F'(A) \\ \downarrow{\scriptstyle F(\phi)} & & \downarrow{\scriptstyle F'(\phi)} \\ F(B) & \xrightarrow{\alpha(B)} & F'(B) \end{array}$$

commutes for every morphism $\phi : A \to B$ (resp. $\phi : B \to A$) in $\mathcal{C}$. The natural transformation α is called *isomorphism of functors* if $F(A) \to F'(A)$ is an isomorphism for every object A of $\mathcal{C}$.

Functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$, together with natural transformations, form a category.

A functor $F : \mathcal{C} \to \mathcal{C}'$ is called *faithful* (resp. *full*, resp. *fully faithful*) if the maps:

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$$

are injective (resp. surjective, resp. bijective).

A functor $F : \mathcal{C} \to \mathcal{C}'$ is called *essentially surjective* if every object of $\mathcal{C}'$ is isomorphic to some $F(A)$ with A an object of $\mathcal{C}$.

A functor $F : \mathcal{C} \to \mathcal{C}'$ is called *equivalence of categories* if there exists a functor $G : \mathcal{C}' \to \mathcal{C}$ such that $F \circ G$ is isomorphic to $Id_{\mathcal{C}'}$ and $G \circ F$ is isomorphic to $Id_{\mathcal{C}}$.

One can prove that a functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if, and only if it is fully faithful and essentially surjective.

We make next a construction of a category, called the *comma category*. Consider the following setting:

$$\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B} ,$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are categories, F,G are functors. We define the *comma category* $(F \downarrow G)$ as follows:

- its objects are triples (A,B,f) such that A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$ and $f : F(A) \to G(B)$ is a morphism in $\mathcal{C}$;
- a morphism between (A,B,f) and (A',B',f') is a pair (g,h) where $g : A \to A'$ and $h : B \to B'$ are morphisms such that the following diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(g)} & F(A') \\ \downarrow f & & \downarrow f' \\ G(B) & \xrightarrow{G(h)} & G(B') \end{array}$$

  commutes.

As a special case, we take F to be the identity functor on $\mathcal{C}$. We fix an object A of $\mathcal{C}$ and define G to be the functor that sends every object of $\mathcal{B}$ to A and every morphism in $\mathcal{B}$ to Id_A. The resulting category is known as the *slice category* and is denoted by $(\mathcal{C} \downarrow A)$ or $\mathcal{C}/A$. Its objects are pairs (B,f) where $f : B \to A$, and a morphism between (B,f) and (B',f') is a morphism $g : B \to B'$ such that $f = f' \circ g$.

1.2 Limits of functors

By a diagram in category $\mathcal{C}$, we mean a functor $F : \mathcal{J} \to \mathcal{C}$, where $\mathcal{J}$ can be thought of as an index category. Let N be an object of $\mathcal{C}$. A *cone* from N to F consists of a family of morphisms (indexed by the objects of $\mathcal{J}$)

$$\phi_i : N \to F(i)$$

such that for any $f_{i,j} : i \to j$ in $\mathcal{J}$ we have

$$F(f_{i,j}) \circ \phi_i = \phi_j.$$

By abuse of language, we call N a cone to F. A *projective limit* of a functor $F : \mathcal{J} \to \mathcal{C}$ is a universal cone to F; that is, a cone N such that any another cone P to F factors uniquely through N.

By taking the dual of all these definitions (cone, universal cone) we can define a cone from F and *injective limits*, which is the dual notion of *projective limit*. For more details, see [8].

Products and *coproducts* are examples of projective and injective limits, respectively, where $\mathcal{J}$ is taken to be a discrete category (category with only identities as morphisms).

1.3 Abelian categories

A category $\mathcal{C}$ is called *additive* if all the sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ have structures of abelian groups such that the composition maps are bi-additive, and if every finite family of objects (i.e. diagram indexed by a finite category) of $\mathcal{C}$ admit coproducts.

A functor F from an additive category $\mathcal{C}$ to an additive category $\mathcal{C}'$ is called *additive* if the maps:

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}'}(F(A), F(B))$$

are homomorphisms of abelian groups for all objects A, B of $\mathcal{C}$.

Assuming the reader is familiar with the definition of exactness in the category **Ab** of abelian groups, we define now exactness in an arbitrary additive category.

- A sequence
$$0 \longrightarrow A \longrightarrow B \xrightarrow{f} C$$
in an additive category $\mathcal{C}$ is exact if the corresponding sequence of abelian groups
$$0 \longrightarrow \mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(X, C)$$
is exact for every object X of $\mathcal{C}$, in which case A is called *kernel* of f. We have a natural morphism $i : \ker(f) \to B$.
- A sequence
$$A \xrightarrow{g} B \longrightarrow C \longrightarrow 0$$
in an additive category $\mathcal{C}$ is exact if the corresponding sequence of abelian groups
$$0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$$
is exact for every object X of $\mathcal{C}$, in which case C is called *cokernel* of g. We have a natural morphism $\pi : B \to \mathrm{coker}(g)$.

Remark 1.3.1. *Kernels* and *cokernels* can also be defined to be projective and injective limits, respectively, where $\mathcal{J}$ is the category consisting of two objects 0 and 1 such that $\mathrm{Hom}(0, 0) = \{\mathrm{Id}_0\}$ $\mathrm{Hom}(0, 1) = \{a, b\}$ $\mathrm{Hom}(1, 0) = \emptyset$ $\mathrm{Hom}(1, 1) = \{\mathrm{Id}_1\}$

Let $f : A \to B$ in an additive category $\mathcal{C}$, then we have

$$\ker f \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{\pi} \mathrm{coker}(f)\ .$$

The *coimage* of f is $\mathrm{coim}(f) := \mathrm{coker}(i)$ and the image of f is $\mathrm{im}(f) := \ker(\pi)$.

Let $f : A \to B$ be a morphism in an additive category $\mathcal{C}$, and suppose that f has an image and a coimage. There exists a unique morphism

$$\bar{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$$

such that the composition

$$A \longrightarrow \mathrm{coim}(f) \xrightarrow{\bar{f}} \mathrm{im}(f) \longrightarrow B$$

is equal to f.

An *abelian category* $\mathcal{C}$ is an additive category in which every morphism $f : A \to B$ in $\mathcal{C}$ has a kernel and a cokernel and such that the morphism $\bar{f} : \mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism. Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ be an abelian category. The category of functors from $\mathcal{C}$ to $\mathcal{C}'$ is an abelian category.

1.4 Complexes in abelian category

Let $\mathcal{C}$ be an abelian category.

- A *cochain complex* $A^\bullet$ in $\mathcal{C}$ is a family of objects $(A^i)_{i\in\mathbb{Z}}$ of $\mathcal{C}$ together with morphisms $d^i \in \mathrm{Hom}_{\mathcal{C}}(A^i, A^{i+1})$, called *coboundary maps*, such that $d^{i+1} \circ d^i = 0$ for every $i \in \mathbb{Z}$.
- Let $A^\bullet$ and $B^\bullet$ be complexes in $\mathcal{C}$. A *morphism of complexes* $f^\bullet : A^\bullet \to B^\bullet$ is a family of morphisms $f^i : A^i \to B^i$ that commute with the coboundary maps d^i for every $i \in \mathbb{Z}$.
- The *i-th cohomology object* of a complex $A^\bullet$ is defined by

$$h^i(A^\bullet) := ker(d^i)/imd^{i-1}.$$

Any morphism $f^\bullet : A^\bullet \to B^\bullet$ of cochain complexes induce morphisms on the corresponding i-th cohomology objects $h^i(f^\bullet) : h^i(A^\bullet) \to h^i(B^\bullet)$

Complexes in $\mathcal{C}$, together with their morphisms, form an abelian category.

Two morphisms of complexes $f^\bullet, g^\bullet : A^\bullet \to B^\bullet$ are homotopic if there exist morphisms $k^i : A^i \to B^{i-1}$ for each i, such that

$$f^i - g^i = e^{i-1}k^i + k^{i+1}d^i$$

where d^i and e^i denote the coboundary maps for the cochain complexes $A^\bullet$ and $B^\bullet$ respectively.

Homotopic morphisms of complexes induce the same morphism on each cohomology object.

1.5 Injective resolutions and derived functors

- An additive covariant (resp. contravariant) functor $F : \mathcal{C} \to \mathcal{C}'$ is called *left exact* if given a short exact sequence in $\mathcal{C}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

the corresponding sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

(resp.

$$0 \longrightarrow F(C) \longrightarrow F(B) \longrightarrow F(A)\ )$$

is an exact sequence in $\mathcal{C}'$.

- An additive covariant (resp. contravariant) functor $F : \mathcal{C} \to \mathcal{C}'$ is called *right exact* if given a short exact sequence in $\mathcal{C}$

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

the corresponding sequence

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$$

(resp.

$$F(C) \longrightarrow F(B) \longrightarrow F(A) \longrightarrow 0\ )$$

is an exact sequence in $\mathcal{C}'$.

- An additive covariant functor is called *exact* if it is both right and left exact.

Let $\mathcal{C}$ be an abelian category.

- An object I of $\mathcal{C}$ is called *injective* if the functor

$$\mathrm{Hom}_{\mathcal{C}}(.,I) : \mathcal{C} \to \mathbf{Ab}$$

is exact.

- An *injective resolution* of an object A of $\mathcal{C}$ consists of a complex $I^\bullet$ of injective objects of $\mathcal{C}$ together with a morphism $\epsilon : A \to I^0$, such that the sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is exact.

An abelian category $\mathcal{C}$ is said to *have enough injectives* if every object of $\mathcal{C}$ is isomorphic to a subobject of an injective object of $\mathcal{C}$.

If an abelian category $\mathcal{C}$ has enough injectives, then each of its objects has an injective resolution.

Injective resolutions of an object A in an abelian category $\mathcal{C}$ are homotopic.

The *right derived functors* of a covariant left-exact functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ has enough injectives are constructed as follow:

- Take an object X of $\mathcal{C}$;

- Construct an injective resolution of X

$$0 \longrightarrow X \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

- Apply F to the resolution, and omit the first term, to get a complex

$$0 \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow \cdots \tag{1}$$

- The *i-th right derived functor of* F is the i-th cohomology of the complex (1).

Notice that in this construction, any injective resolution of X can be chosen since they are homotopic.

1.6 Adjoint functors

Let $F : \mathcal{C} \to \mathcal{C}'$ be a covariant additive functor. Then $G : \mathcal{C}' \to \mathcal{C}$ is said to be a *left adjoint* to F if, for any two objects A of $\mathcal{C}'$ and B of $\mathcal{C}$ we have isomorphisms of abelian groups

$$\mathrm{Hom}_{\mathcal{C}'}(A, F(B)) \simeq \mathrm{Hom}_{\mathcal{C}}(G(A), B)$$

which are functorial in A and B. If this is the case, F is called *right adjoint* of G.

If it exists, an adjoint functor to a given functor is unique up to a unique isomorphism.

Proposition 1.6.1. *Let $G : \mathcal{C} \to \mathcal{C}'$ be an additive covariant functor. If G admits a left adjoint F, then G is left exact.*

Proof. Let

$$0 \to A' \to A \to A''$$

be an exact sequence in $\mathcal{C}$. Then, since Hom is a left exact functor, we have that, for every object X of $\mathcal{C}$, the sequence

$$0 \to \mathrm{Hom}(X, A') \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, A'')$$

is exact. In particular, for every object B of $\mathcal{C}'$ the sequence

$$0 \to \mathrm{Hom}(F(B), A') \to \mathrm{Hom}(F(B), A) \to \mathrm{Hom}(F(B), A'')$$

is exact. But, since F is a left adjoint to G, the above sequence is the same as

$$0 \to \mathrm{Hom}(B, G(A')) \to \mathrm{Hom}(B, G(A)) \to \mathrm{Hom}(B, G(A''))$$

which is then exact, and therefore G is left exact. □

Proposition 1.6.2. *With the settings of Proposition 1.6.1, if F is exact, then G sends injective objects in $\mathcal{C}$ to injective objects in $\mathcal{C}'$.*

Proof. Let I be an injective object of $\mathcal{C}$, then $\mathrm{Hom}(., I)$ is exact in $\mathcal{C}$. We want to prove that $\mathrm{Hom}(., G(I))$ is exact in $\mathcal{C}'$. But this is immediate since $\mathrm{Hom}(., G(I)) = \mathrm{Hom}(F(.), I)$, F is exact and I is injective in $\mathcal{C}$. □

2 Sheaf cohomology on sites

In this section, we will begin by defining sites and continuous functions between them. We will then define the category of sheaves on a site and see how continuous functions between sites induce functors between the corresponding categories of sheaves. These functors are left exact, and sheaf cohomology functors are basically defined as their right derived functors.

2.1 The category of sites

Let X be a topological space. Denote by T the topology on X, i.e. the family of open sets of X. Then, T can be viewed as a category if we define, for U, V in T:

$$\mathrm{Hom}_T(U,V) = \begin{cases} \emptyset & \text{if } U \text{ is not a subset of } V, \\ \{U \hookrightarrow V\} & \text{if } U \subseteq V. \end{cases}$$

The global space X is the final object of the category T. The product of finitely many objects of T is their intersection, and the coproduct of arbitrarily many open sets of T is their union.

Grothendieck's generalization of a topology consists of replacing the category of open sets of a topological space by any category and attaching to it a set of coverings for each of its objects. But first, we need a "substitute" for the notion of intersection. In an abstract category, fibered products play this role.

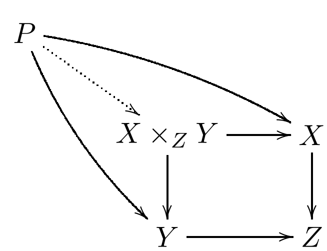
Definition 2.1.1. (Fibered products) Let $\mathcal{C}$ be a category that admits finite limits, and consider the following diagram in $\mathcal{C}$

$$X \longrightarrow Z \longleftarrow Y$$

The *fibered product of* X *and* Y *over* Z is the projective limit of the above diagram. We denote it by $X \times_Z Y$. It is equipped with two canonical projections:

$$X \times_Z Y \to X$$
$$X \times_Z Y \to Y$$

satisfying the following universal property: For any object P of $\mathcal{C}$ together with morphisms $P \to X$ and $P \to Y$, there exists a unique morphism $P \to X \times_Z Y$ making the following diagram :



commutative.

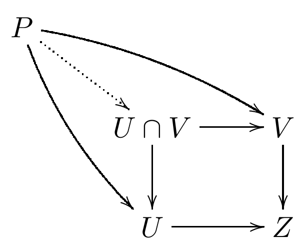
Definition 2.1.2. (Grothendieck topology) Let $\mathcal{C}$ be a category that admits finite limits. A *Grothendieck topology* on $\mathcal{C}$ is an assignement to each object U of $\mathcal{C}$ a set of coverings $\mathbf{cov}(U)$ such that:

- if $\{V \to U\}$ is an isomorphism in $\mathcal{C}$, then it is in $\mathbf{cov}(U)$;
- if $\{U_i \to U\}$ is in $\mathbf{cov}(U)$ and $V \to U$ is a morphism in $\mathcal{C}$, then $\{V \times_U U_i \to V\}$ is in $\mathbf{cov}(V)$;
- if $\{U_i \to U\}$ is in $\mathbf{cov}(U)$ and for every i $\{U_{ij} \to U_i\}$ is in $\mathbf{cov}(U_i)$ then $\{U_{ij} \to U_i \to U\}$ (obtained via composition) is in $\mathbf{cov}(U)$.

A *site* X consists of a category $\mathcal{C}_X$ having finite limits, together with a Grothendieck topology. $\mathcal{C}_X$ will be called the *underlying category* of X.

The underlying categories of the sites we consider throughout this work are assumed to admit terminal objects.

Example 2.1.3. (Site assigned to a topological space) Let X be a topological space, and $\mathrm{Op}(X)$ be the category whose objects are the open sets of X and morphisms are simply the inclusions between two open sets of X. Define a Grothendieck topology on $\mathrm{Op}(X)$ by assigning to each open set U of X (i.e. object of $\mathrm{Op}(X)$) the collection $\{U_i \subseteq U\}$ where the U_i are an open cover of U, in the usual sense, that is, $\cup_{i\in I} U_i = U$. To see this, let us first prove that the fibered product of two open sets U and V over some set Z containing them both is their intersection. If there exists $P \subseteq U \subseteq Z$ and $P \subseteq V \subseteq Z$, then $P \subseteq U \cap V$, that is, there exists a unique morphism in $\mathrm{Op}(X)$ from P to $U \cap V$ making the diagram



commutative. So, $U \times_Z V$ is indeed the same as $U \cap V$. The verification of the axioms for a Grothendieck topology is straightforward.

Example 2.1.4. (The global classical topology) Consider the category **Top** of topological spaces. Define a Grothendieck topology on **Top** by assigning to each topological space X the collection $\{X_i \to X\}$ of open continuous injective maps such that the union of their images covers X.

Example 2.1.5. (The global Zariski Topology on a scheme) The same as the global classical topology by taking specifically the schemes with the Zariski topology as objects.

Definition 2.1.6. (Continuous functions on sites) Let X_1 and X_2 be sites. A *continuous function* $f : X_1 \to X_2$ consists of a covariant functor $f^c : \mathcal{C}_{X_2} \to \mathcal{C}_{X_1}$ that preserves terminal objects, fibered products and coverings.

Proposition 2.1.7. *Sites, together with continuous functions, form a category.*

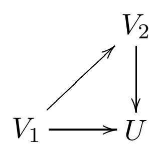
Proof. The identity continuous function on a site X is the obvious identity functor $Id_{\mathcal{C}_X}$. The composition of continuous functions $X_1 \to X_2 \to X_3$ is the composition of functors $\mathcal{C}_{X_3} \to \mathcal{C}_{X_2} \to \mathcal{C}_{X_1}$. Fibered products of $\mathcal{C}_{X_3}$ are sent via the first arrow to fibered products of $\mathcal{C}_{X_2}$ which, in their turn, are sent via the second arrow to fibered products of $\mathcal{C}_{X_1}$; hence the composition sends fibered products to fibered products. Similarly, the composition, as defined, sends coverings in X_3 to coverings in X_1, and preserves terminal objects. So, it is a continuous function. ☐

We will denote this category by **Sit**.

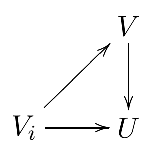
Notice that given a continuous map $f : X \to Y$ where X and Y are topological spaces, and f is continuous in the usual sense, the functor $f^c : \mathrm{Op}(Y) \to \mathrm{Op}(X)$ defined by sending an open set U of Y to its pre-image $f^{-1}(U)$ (which is open in X since f is continuous in the usual sense) is a covariant functor satisfying the conditions of the previous definition, and indeed f is continuous in the sense of sites.

Now, if we suppose we are given a site X, and an object U in $\mathcal{C}_X$, then we define a category $\mathcal{C}_U$, with a Grothendieck topology on it, as follows:

- its objects are morphisms $V \to U$ in $\mathcal{C}_X$;
- its morphisms are commutative diagrams:



- a covering of $V \to U$ in $\mathcal{C}_U$ is defined by a set of morphisms



  where $\{V_i \to V\}$ is a covering of V in $\mathcal{C}_X$.

2.2 Sheaves on sites

We would like to define sheaves of abelian groups on a site, in an analogous way to the definition of sheaves of abelian groups on a topological space.

Definition 2.2.1. A *presheaf* of abelian groups on a site X is a contravariant functor from $\mathcal{C}_X$ to **Ab**:

$$\mathcal{F} : \mathcal{C}_X^{op} \to \mathbf{Ab}.$$

For any U in $Ob(\mathcal{C}_X)$, the elements of $\mathcal{F}(U)$ will be called *sections*. If $V \to U$ is a given morphism in $\mathcal{C}_X$, then the image of an element s of $\mathcal{F}(U)$ under $\mathcal{F}(U) \to \mathcal{F}(V)$ will be called the *restriction* *of s to V* and will be denoted by $s\mid_V$. A presheaf $\mathcal{F}$ on X is a *sheaf* if for every object U, with a covering $\{U_i \to U\}$, the following sequence:

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \tag{2}$$

is exact. This means that if $s \in \mathcal{F}(U)$ is such that $s\mid_{U_i} = 0$ for every $i \in I$ then $s = 0$, and that the image of $\mathcal{F}(U)$ under the first arrow is equal to the equalizer of the double arrow, that is, if we have $s_i \in \mathcal{F}(U_i)$ for each i, such that $s_i\mid_{U_i \times_U s_j} = s_j\mid_{U_i \times_U U_j}$ then there exists $s \in \mathcal{F}(U)$ such that $s\mid_{U_i} = s_i$ for every $i \in I$.

From now on, by presheaf on a site X, we will mean a presheaf of abelian groups on X, and by sheaf on X, we will mean a sheaf of abelian groups on X.

Definition 2.2.2. A *morphism of presheaves* is a natural transformation of contravariant functors.

We define for a site X *the category of presheaves on* X, that we denote by $\mathbf{Presh}(X)$, to be the category whose objects are presheaves on X and whose morphisms are morphisms of presheaves. It is an abelian category that has enough injectives (cf. [11], section I.2.1).

Let $f : X \to Y$ be a continuous function of sites (we will denote by f^c the corresponding functor from $\mathcal{C}_Y$ to $\mathcal{C}_X$), and let $\mathcal{G}'$ be a presheaf on X. We define a presheaf on Y by:

$$f_p\mathcal{G}'(U) = \mathcal{G}'(f^c(U)) \text{ for any } U \in \mathcal{C}_Y,$$

and if we have a morphism of presheaves $a' : \mathcal{G}' \to \mathcal{H}'$ on X, we define a morphism of presheaves on Y, $f_pa' : f_p\mathcal{G}' \to f_p\mathcal{H}'$ by $f_pa'(U) = a'(f_p(U))$ for any object U of $\mathcal{C}_Y$. This gives a functor

$$f_p : \mathbf{Presh}(X) \to \mathbf{Presh}(Y)$$

Proposition 2.2.3. *If $\mathcal{G}'$ is a sheaf, then so is $f_p\mathcal{G}'$.*

Proof. Since f is a continuous function of sites, then it maps fibered products of X to fibered products of Y and coverings of X to coverings of Y. Since $\mathcal{G}'$ is a sheaf on X then the sequence:

$$\mathcal{G}'(f^c(U)) \to \prod_i \mathcal{G}'(f^c(U_i)) \rightrightarrows \prod_{i,j} \mathcal{G}'(f^c(U_i) \times_X f^c(U_j))$$

is exact. But this is the same as the sequence:

$$f_p\mathcal{G}'(U) \to \prod_i f_p\mathcal{G}'(U_i) \rightrightarrows \prod_{i,j} f_p\mathcal{G}'(U_i \times_Y U_j)$$

which is then exact, as desired. ∎

Proposition 2.2.4. *The functor $f_p : \mathbf{Presh}(X) \to \mathbf{Presh}(Y)$ has a left adjoint, that we will denote by f^p.*

Proof. We have to show that for every $\mathcal{F} \in \mathbf{Presh}(Y)$, there exists a presheaf $f^p\mathcal{F}$ in $\mathbf{Presh}(X)$ and for each $\mathcal{G}'$ in $\mathbf{Presh}(X)$, we have an isomorphism of abelian groups

$$\mathrm{Hom}(f^p\mathcal{F}, \mathcal{G}') \simeq \mathrm{Hom}(\mathcal{F}, f_p\mathcal{G}')$$

which is functorial in $\mathcal{G}'$. So, let $\mathcal{F}$ be a presheaf on Y. We want to define $f^p\mathcal{F}(U')$ for every U' in $\mathcal{C}_X$.

First, we consider all pairs $(U,\phi^{'})$, with U an object in $\mathcal{C}_Y$ and $\phi^{'} : U' \to f^c(U)$ a morphism in $\mathcal{C}_X$. We define a morphism of pairs $(U_1, \phi_1^{'}) \to (U_2, \phi_2^{'})$ to be a morphism $\phi : U_1 \to U_2$ such that the diagram

$$\begin{array}{ccc} & & f^c(U_1) \\ & \overset{\phi_1^{'}}{\nearrow} & \big\downarrow f^c(\phi) \\ U' & \xrightarrow{\phi_2^{'}} & f^c(U_2) \end{array}$$

commutes. These pairs, with the above defined morphisms, form a category $\mathcal{I}_{U'}$, and we have a contravariant functor

$$\mathcal{F}_{U'} : \mathcal{I}_{U'} \to \mathbf{Ab}$$

that sends a pair $(U,\phi^{'})$ to the abelian group $\mathcal{F}(U)$. We define, for every U' in $\mathcal{C}_X$,

$$f^p\mathcal{F}(U') := \varinjlim_{\mathcal{I}_{U'}} \mathcal{F}_{U'} = \varinjlim_{(U,\phi^{'})} \mathcal{F}(U).$$

If we have a morphism $\alpha^{'} : U' \to V'$ in $\mathcal{C}_X$, it induces a functor $\mathcal{I}_{V'} \to \mathcal{I}_{U'}$ defined by mapping the pair $(V,\phi^{'} : V' \to f^c(V))$ to the pair $(V,\phi^{'} \circ \alpha^{'} : U' \to f^c(V))$. This gives us a homomorphism

$$\varinjlim_{\mathcal{I}_{V'}} \mathcal{F}_{V'} \to \varinjlim_{\mathcal{I}_{U'}} \mathcal{F}_{U'},$$

and hence a homomorphism

$$f^p\mathcal{F}(V') \to f^p\mathcal{F}(U').$$

This means that $f^p\mathcal{F}$ is a presheaf on X.

Now, we want to show adjointness, which means that we want to show that we have isomorphisms

$$\mathrm{Hom}(f^p\mathcal{F}, \mathcal{G}') \simeq \mathrm{Hom}(\mathcal{F}, f_p\mathcal{G}')$$

which are functorial in $\mathcal{G}'$. So, let $v : f^p\mathcal{F} \to \mathcal{G}'$ be a morphism of presheaves on X. For every $U \in \mathcal{C}_Y$, we get a homomorphism

$$v(f^c(U)) : f^p\mathcal{F}(f^c(U)) \to \mathcal{G}'(f^c(U)) = f_p\mathcal{G}'(U). \tag{3}$$

Now, the pair $(U, \mathrm{Id}_{f^c(U)})$ is an object of the category $\mathcal{I}_{f^c(U)}$ since $\mathcal{I}_{f^c(U)}$ consists of pairs (V,ψ), where V is an object of $\mathcal{C}_Y$ and ψ is a morphism from $f^c(U)$ to $f^c(V)$. Hence, by the properties of inductive limits, we have a canonical homomorphism

$$\mathcal{F}(U) = \mathcal{F}_{f^c(U)}(U, \mathrm{Id}_{f^c(U)}) \to \varinjlim_{\mathcal{I}_{f^c(U)}} \mathcal{F}_{\mathcal{I}_{f^c(U)}} = f^p\mathcal{F}(f^c(U)). \tag{4}$$

Composing the homomorphisms in (3) and (4), we get a homomorphism

$$\mathcal{F}(U) \to f_p\mathcal{G}'(U).$$

Since U was chosen arbitrarily, we get a morphism of presheaves on Y, $w : \mathcal{F} \to f_p\mathcal{G}'$. Consequently, we have a homomorphism of abelian groups

$$\mathrm{Hom}(f^p\mathcal{F}, \mathcal{G}') \to \mathrm{Hom}(\mathcal{F}, f_p\mathcal{G}')$$

which is functorial in $\mathcal{G}'$. It sends $v : f^p\mathcal{F} \to \mathcal{G}'$ to $w : \mathcal{F} \to f_p\mathcal{G}'$, as shown above. Conversely, let $u : \mathcal{F} \to f_p\mathcal{G}'$ be a morphism of presheaves on Y, and let U' be an object of $\mathcal{C}_X$. For every pair $(U,\phi^{'})$ of $\mathcal{I}_{U'}$, we get a homomorphism

$$\mathcal{F}_{U'}(U,\phi^{'}) = \mathcal{F}(U) \to f_p\mathcal{G}'(U) = \mathcal{G}'(f^c(U)) \to \mathcal{G}'(U')$$

which is functorial in $(U,\phi^{'})$, and where the first arrow is $u(U)$ and the second is $\mathcal{G}'(\phi^{'})$. By the universality of inductive limits, we get a homomorphism

$$f^p\mathcal{F}(U') = \varinjlim_{\mathcal{I}_{U'}} \mathcal{F}_{U'} \to \mathcal{G}'(U')$$

which is functorial in U'. So, this gives a morphism of presheaves on X

$$t : f^p\mathcal{F} \to \mathcal{G}'.$$

This leads a homomorphism of abelian groups

$$\mathrm{Hom}(\mathcal{F}, f_p\mathcal{G}') \to \mathrm{Hom}(f^p\mathcal{F}, G')$$

which is functorial in $\mathcal{G}'$. It sends $u : \mathcal{F} \to f_p\mathcal{G}'$ to $t : f^p\mathcal{F} \to \mathcal{G}'$. One can check that $(v \mapsto w)$ and $(u \mapsto t)$ are inverses of each other, hence we have an isomorphism of abelian groups

$$\mathrm{Hom}(f^p\mathcal{F}, G') \simeq \mathrm{Hom}(\mathcal{F}, f_p\mathcal{G}')$$

which is functorial in $\mathcal{G}'$, as desired. □

We define *the category of sheaves on* X, that we denote by $\mathbf{Sh}(X)$, to be the full subcategory of $\mathbf{Presh}(X)$, whose objects are the sheaves on X. So, for every site X, we have a fully faithful functor

$$i_X : \mathbf{Sh}(X) \to \mathbf{Presh}(X) \quad (5)$$

Theorem 2.2.5. *(i) For every site X, the functor $i_X : \mathbf{Sh}(X) \to \mathbf{Presh}(X)$ has a left adjoint $i_X^{ad} : \mathbf{Presh}(X) \to \mathbf{Sh}(X)$ (so i_X is left exact). Moreover, i_X^{ad} is exact.*

(ii) For every site X, the category $\mathbf{Sh}(X)$ is an abelian category that has enough injectives.

Proof. See [11], sections I.3.1 and I.3.2. □

Now, let $f : X \to Y$ be a continuous function of sites, then we have

$$f_p : \mathbf{Presh}(X) \to \mathbf{Presh}(Y)$$

that has a left adjoint

$$f^p : \mathbf{Presh}(Y) \to \mathbf{Presh}(X),$$

and

$$i_X : \mathbf{Sh}(X) \to \mathbf{Presh}(X)$$

that has a left adjoint

$$i_X^{ad} : \mathbf{Presh}(X) \to \mathbf{Sh}(X),$$

and

$$i_Y : \mathbf{Sh}(Y) \to \mathbf{Presh}(Y)$$

that has a left adjoint

$$i_Y^{ad} : \mathbf{Presh}(Y) \to \mathbf{Sh}(Y).$$

We define a functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ by

$$f_* = i_Y^{ad} \circ f_p \circ i_X.$$

But by Proposition 2.2.3, we have that for any sheaf $\mathcal{F}$ on X, $f_p(i_X(\mathcal{F}))$ is a sheaf, so we get that $f_* = f_p \circ i_X$. We also define a functor $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$ by

$$f^* = i_X^{ad} \circ f^p \circ i_Y.$$

We will prove that f_* and f^* form a pair of adjoint functors, but we need first the following lemma.

Lemma 2.2.6. *Consider the following commutative diagram of categories*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\gamma} & \mathcal{B} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta} \\ \mathcal{C} & \xrightarrow{\beta} & \mathcal{D} \end{array}$$

where δ is fully faithful, and α, β admit left adjoints $\tilde{\alpha}$, $\tilde{\beta}$, respectively. Then $\tilde{\alpha} \circ \tilde{\beta} \circ \delta$ is the left adjoint of γ.

Proof. We have

$$\begin{aligned} \mathrm{Hom}(\tilde{\alpha}\tilde{\beta}\delta(.), .) &= \mathrm{Hom}(\tilde{\beta}\delta(.), \alpha(.)) \\ &= \mathrm{Hom}(\delta(.), \beta\alpha(.)) \\ &= \mathrm{Hom}(\delta(.), \delta\gamma(.)) \\ &= \mathrm{Hom}(., \gamma(.)) \end{aligned}$$

where the two first equalities come from the fact that $\tilde{\alpha}$, $\tilde{\beta}$ are left adjoints to α, β, respectively, the third comes from commutativity, and the last comes from the full faithfulness of δ. □

Proposition 2.2.7. *Let $f : X \to Y$ be a continuous function of sites and let f_* and f^* be the functors defined above, then we have:*

(i) The functor f^* is a left adjoint to f_*, and hence f_* is left exact.

(ii) If $\mathcal{C}_X$ and $\mathcal{C}_Y$ have terminal objects, sent one to the other under f, then f^* is exact (and hence f_* preserves injectives).

Proof. (i) We apply Lemma 2.2.6 to the commutative diagram

$$\begin{array}{ccc} \mathbf{Sh}(X) & \xrightarrow{f_*} & \mathbf{Sh}(Y) \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ \mathbf{Presh}(X) & \xrightarrow{f_p} & \mathbf{Presh}(Y) \end{array}$$

(ii) See [11], section I.3.6.

□

Proposition 2.2.8. *Let* **Cat** *denote the category of small categories. The map $\underline{\mathrm{Sh}} : \mathbf{Sit} \to \mathbf{Cat}$ defined by:*

$$\underline{\mathrm{Sh}}(X) := \mathbf{Sh}(X)$$
$$\underline{\mathrm{Sh}}(f : X \to Y) := (f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y))$$

is a covariant functor.

Proof. The only non-trivial thing to check is that given $g \circ f : X \to Y \to Z$, we get $\underline{\mathrm{Sh}}(g \circ f) = \underline{\mathrm{Sh}}(g) \circ \underline{\mathrm{Sh}}(f)$, which is the same as checking that , $(g \circ f)_* = g_* \circ f_*$. So, let $\mathcal{F}$ be a sheaf on X, W be an object of $\mathcal{C}_Z$, we have that

$$(g \circ f)_*\mathcal{F}(W) = \mathcal{F}(g^c \circ f^c(W)) = \mathcal{F}(g^c(f^c(W)) = g_*\mathcal{F}(f^c(W)) = g_*(f_*\mathcal{F}(W)) = (g_* \circ f_*)\mathcal{F}(W)$$

as desired. □

Example 2.2.9. Suppose we are given a site X. We construct a site X^P as follows:

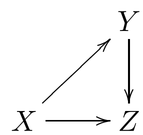
- $\mathcal{C}_{X^P} := \mathcal{C}_X$
- for every $U \in Y$, $\mathbf{cov}(U) := \{\text{isomorphisms } \{V \to U\}\}$

If $\mathcal{F}$ is a presheaf on X, then it is a sheaf on X^P, since the only coverings of X^P are the trivial ones. Hence $\mathbf{Presh}(X) = \mathbf{Sh}(X^P)$. Let $i : X \to X^P$ be the continuous function defined by sending every object of C_{X^P} to itself in C_X (indeed it is a continuous function because coverings of X^P will be sent to coverings of X, and clearly the function preserves fibered products). Then i induces a functor $i_* : \mathbf{Sh}(X) \to \mathbf{Sh}(X^P)$, which is exactly the functor i_X from **Sh**(X) to **Presh**(X) (cf. 5), since $\mathbf{Presh}(X) = \mathbf{Sh}(X^P)$.

2.3 Sheaf cohomology

We fix an object Z of **Sit**. We consider the pairs $(X \to Z, \mathcal{F})$ where X is an object of **Sit**, and $\mathcal{F}$ is a sheaf of abelian group on X. A morphism between two such pairs $(X \to Z, \mathcal{F})$ and $(Y \to Z, \mathcal{G})$ is a pair (ϕ, α) consisting of:

- A continuous function of sites $\phi : X \to Y$ such that that the diagram:

  commutes;
- A morphism of sheaves $\alpha : \mathcal{G} \to \phi_* \mathcal{F}$.

Proposition 2.3.1. *The pairs defined above, together with their morphisms, form a category* $\mathbf{ShS}_Z$*.*

Proof. The identity morphism on a pair $(X \to Z, \mathcal{F})$ is $(\mathrm{Id}_X, \mathrm{Id}_{\mathcal{F}})$. Next, given

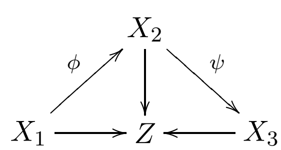
$$(\phi, \alpha) : (X_1 \to Z, \mathcal{F}_1) \to (X_2 \to Z, \mathcal{F}_2)$$

$$(\psi, \beta) : (X_2 \to Z, \mathcal{F}_2) \to (X_3 \to Z, \mathcal{F}_3),$$

we define the composition

$$(\psi, \beta) \circ (\phi, \alpha) := (\psi \circ \phi, \psi_*(\alpha) \circ \beta).$$

Indeed, the diagram



is commutative, since each of its parts is commutative.

And,

$$\psi_*(\alpha) \circ \beta : \mathcal{F}_3 \to \psi_* \mathcal{F}_2 \to (\psi_* \circ \phi_*) \mathcal{F}_1.$$

But since $\psi_* \circ \phi_* = (\psi \circ \phi)_*$ then we have the desired result. □

For every object $(f : X \to Z, \mathcal{F})$ of $\mathbf{ShS}_Z$, f_* is left exact, we denote its right derived functors by $R^i f_*$, $i \in \mathbb{Z}$, $i \geq 0$. So we get a family $R^i f_*(\mathcal{F})$ of sheaves on Z, and we have

Proposition 2.3.2. *The map*

$$(f : X \to Z, \mathcal{F}) \mapsto R^i f_* \mathcal{F}$$

from $\mathbf{ShS}_Z$ *to* $\mathbf{Sh}(Z)$ *is a contravariant functor, for each* $i \geq 0$*.*

Proof. Let $(h, \alpha) : (f : X \to Z, \mathcal{F}) \longrightarrow (g : Y \to Z, \mathcal{G})$ be a morphism in $\mathbf{ShS}_Z$. Consider an injective resolution of $\mathcal{G}$ in $\mathbf{Sh}(Y)$:

$$0 \to \mathcal{G} \to \mathcal{I}^\bullet,$$

and an injective resolution of $\mathcal{F}$ in $\mathbf{Sh}(X)$:

$$0 \to \mathcal{F} \to \mathcal{J}^\bullet.$$

Applying g_* and f_* respectively to the above resolutions yields complexes

$$0 \to g_*\mathcal{G} \to g_*\mathcal{I}^\bullet,$$

and

$$0 \to f_*\mathcal{F} \to f_*\mathcal{J}^\bullet.$$

where $g_*\mathcal{I}^\bullet$ and $f_*\mathcal{J}^\bullet$ are complexes of injective objects, since g_* and f_* preserve injectives. Now, if we apply g_* to the morphism $\alpha : \mathcal{G} \to h_*\mathcal{F}$ we get a morphism $g_*(\alpha) : g_*\mathcal{G} \to f_*\mathcal{F}$, since $g_* \circ h_* = f_*$. This gives a morphism $g_*\mathcal{G} \to f_*\mathcal{J}^\bullet$ obtained via the composition $g_*\mathcal{G} \to f_*\mathcal{F} \to f_*\mathcal{J}^\bullet$, and since $f_*\mathcal{J}^\bullet$ is a complex of injective objects, we obtain a morphism of complexes $g_*\mathcal{I}^\bullet \to f_*\mathcal{J}^\bullet$, which is unique up to homotopy, and consequently a morphism between each of its cohomology objects, that is, a morphism $R^i g_*(\mathcal{G}) \to R^i f_*(\mathcal{F})$ for every $i \geq 0$. □

In particular, take Z to be the punctual site $\{.\}$, whose category consists of only one object U where U admits the unique trivial covering $U \to U$. Then there exists a unique continuous function from any site X to $\{.\}$, notably the one induced by the functor that sends the unique object of $\{.\}$ to X_{ter}, the terminal object of $\mathcal{C}_X$ (since a continuous function of sites must preserve the terminal object). Denote this function by γ, then we have

$$\gamma_* : \mathbf{Sh}(X) \to \mathbf{Sh}(\{.\}) = \mathbf{Presh}(\{.\}) = \mathbf{Ab}$$

which maps a sheaf $\mathcal{F}$ on X to $\gamma_*\mathcal{F}(.) = \mathcal{F}(\gamma^c(.)) = \mathcal{F}(X_{ter})$.

We call this γ_* the *global section functor*. For a given X, γ_* will be denoted by Γ_X.

Since we have for every site X, a unique morphism from X to the punctual site, then we can denote the objects of $\mathbf{ShS}_{\{.\}}$ simply by $(X,\mathcal{F})$.

Definition 2.3.3. The functors

$$H^i : \mathbf{ShS}_{\{.\}} \to \mathbf{Ab}$$

that map $(X,\mathcal{F})$ to $R^i\Gamma_X(\mathcal{F})$ $(i \geq 0)$ are called the *sheaf cohomology functors*.

Given a continuous function of sites $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, then $(X,\mathcal{F})$ and $(Y,f_*\mathcal{F})$ are objects of $\mathbf{ShS}_{\{.\}}$, and the pair $(f,\mathrm{Id}_{f_*\mathcal{F}})$ is a morphism between them, so by Proposition 2.3.2 we have morphisms

$$H^i(Y, f_*\mathcal{F}) \to H^i(X, \mathcal{F}) \tag{6}$$

for $i \geq 0$.

3 Étale Cohomology

In this section, we define the étale site X_{et} of a locally Noetherian scheme X, and we prove all the requirements to define the sheaf cohomology functors on X_{et}. In other words, we make explicit all the functors we constructed in section 2, but for the case of X_{et}. Throughout this section, by a scheme X, we will mean a locally Noetherian scheme.

3.1 Étale morphisms

Definition 3.1.1.

- A ring homomorphism $A \to B$ is *flat* if the functor $M \mapsto B \otimes_A M$ is exact. Recall that this functor is always right exact, so in other words, flatness means preserving injectivity after tensoring. We also say that B is a *flat A-algebra*.
- A morphism of schemes $f : X \to Y$ is *flat* if for all x in X, the local ring homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

Definition 3.1.2.

- Let A and B be local rings, with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. A homomorphism $\phi : A \to B$ is called *unramified* if $\phi(\mathfrak{m}_A) = \mathfrak{m}_B$ and if the field $B/\mathfrak{m}_B$ is a finite and seperable extension of $A/\mathfrak{m}_A$
- A morphism of schemes $f : X \to Y$ is *locally of finite type* if Y can be covered by open affine schemes $V_i = \mathrm{Spec} A_i$ such that for every i, $f^{-1}(V_i)$ can be covered by open affine schemes $U_{ij} = \mathrm{Spec} B_{ij}$ with B_{ij} being finitely-generated A_i-algebras. If the U_{ij} can be chosen to be finitely many, then we say that f is *of finite type*.
- A morphism of schemes $f : X \to Y$ is *unramified* if it is of finite type and if for all x in X, the local ring homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is unramified.

Definition 3.1.3. A morphism of schemes is *étale* if it is both flat and unramified.

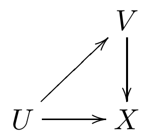
We next list, without proof, some useful properties of étale morphisms. For proofs, see [9], Chapter I, section 3.

- Open immersions are étale.
- The composition of two étale morphisms is again étale.
- Étale morphisms are stable under base change.
- If $g \circ f$ is étale and g is étale, then f is étale.
- An étale morphism is open.
- If $f : X \to Y$ is étale, and Y is reduced (resp. normal, regular), then X is reduced (resp. normal, regular).
- If $f : X \to Y$ is a morphism of finite type, then the set of points where f is étale is open in X.

3.2 The étale site of a scheme

Definition 3.2.1. Let X be a scheme. We define the *small étale site* X_{et} of X as follows:

- its underlying category is the category of étale X-schemes. An object of this category is a scheme U together with an étale morphism from U to X;
- a morphism between $U \to X$ and $V \to X$ is a morphism $U \to V$ making the diagram



  commutative;
- a family of étale morphisms $\{r_i : U_i \to U\}$ is a covering of U if $U = \cup r_i(U_i)$.

There are different versions of the étale site.

- The big étale site (X_{Et}): Its underlying category is **Sch**$/X$, which is the category of X-schemes, whose objects are morphisms from arbitrary schemes U to X, and coverings are surjective families of étale X-morphisms $\{U_i \to U\}$.
- The flat site (X_{fl}): Similar to the big étale topology except for the coverings which are here surjective families of flat and finite type X-morphisms.

Given a Zariski-open set U of X, then $U \hookrightarrow X$ is an open immersion and thus étale, which implies that $U \hookrightarrow X$ is open in the étale topology. This gives a functor ϵ^c from the category of open sets of X_{Zar} to that of X_{et}, which yields a continuous function

$$\epsilon : X_{et} \to X_{Zar}.$$

3.2.1 Sheaves on X_{et}

Let X be a scheme. Sheaves on X_{et} are defined the same way as on any site. By abuse of notation, we will denote $\mathcal{F}(U \to X)$ by $\mathcal{F}(U)$, for every étale $U \to X$.

We will say, for practical reasons, that $\mathcal{F}$ satisfies condition (S) when the sheaf condition sequence (2) is exact (cf. section 2.2).

Notice that the restriction of $\mathcal{F}$ as a sheaf on the étale site to the given étale open set U gives a sheaf on U_{Zar}, i.e., U endowed with the Zariski topology. Again, given a Zariski cover, the sheaf condition given by the sequence is the familiar sheaf condition on Zariski topological spaces. The next proposition gives a simpler criterion for a presheaf on an étale site to be a sheaf.

Proposition 3.2.2. *Let X be a scheme, $\mathcal{F}$ be a presheaf on X_{et} and suppose that:*

- *$\mathcal{F}$ satisfies (S) for Zariski open coverings;*
- *$\mathcal{F}$ satisfies (S) for étale coverings $V \to U$ with both U and V affine.*

Then $\mathcal{F}$ is a sheaf on X_{et}.

Proof. See [9], chapter II, 1.5. □

Definition 3.2.3.
- Let X be a scheme, and let k be a field. A *k-point* $\bar{x}$ of X is a morphism of schemes Spec $k \to X$. If k is separably closed then, we call it *geometric point.*
- An *étale neighborhood* of a k-point $\bar{x}$ is an étale morphism $U \to X$ together with a k-point $\bar{u} : \text{Spec} k \to U$ above $\bar{x}$ such that the following diagram

$$\begin{array}{ccc} & \nearrow & U \\ & & \downarrow \\ \text{Spec } k & \longrightarrow & X \end{array}$$

commutes.

Definition 3.2.4. Let X be a scheme and $\mathcal{F}$ be a sheaf on X_{et}, and let $\bar{x}$ be a geometric point of X. We define the *stalk* of $\mathcal{F}$ at $\bar{x}$ to be

$$\mathcal{F}_{\bar{x}} := \lim_{\rightarrow} \mathcal{F}(U).$$

where the limit is taken over all étale neighborhoods $(U, \bar{u})$ of $\bar{x}$.

We list next few examples of sheaves.

The structure sheaf on X_{et}

For any étale morphism $U \to X$, define the structure sheaf on X_{et} as follows: $\mathcal{O}_{X_{et}}(U) = \Gamma(U, \mathcal{O}_U)$. It is a sheaf on U endowed with the Zariski topology. So the first condition of Proposition 3.2.2 is satisfied. To see that it is a sheaf on X_{et}, we need the following:

Proposition 3.2.5. *Let $f : A \to B$ be a faithfully flat homomorphism. Then the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{b \to 1\otimes b - b \otimes 1} B \otimes_A B$$

is exact.

Proof. We will proceed with the following argument by Grothendieck: first, we will show that if f admits a section s (that is a map $s : B \to A$ such that $s \circ f = Id_A$), then the statement is true. Let b be in the kernel of the map $B \to B \otimes_A B$, then $1 \otimes b - b \otimes 1 = 0$. We want to find a pre-image of b in A. Let $B \otimes_A B \to B$ be the homomorphism of rings sending $b \otimes b'$ to $b.(f \circ s(b'))$. So g sends $1 \otimes b - b \otimes 1$ to $fs(b) - b$. But $1 \otimes b - b \otimes 1 = 0$ so $b = fs(b) = f(s(b)) \in f(A)$, as desired. Next, if the statement holds for $A' \to A' \otimes_A B$ sending a' to $a' \otimes 1$, where A' is a faithfully flat A-module, then it holds for f, since the sequence for $A' \to A' \otimes_A B$ comes from that of $A \to B$ by tensoring by A', which is faithfully flat. Now, the morphism $B \to B \otimes_A B$ sending b to $b \otimes 1$ admits a section, notably the map $B \otimes_A B \to B$ sending $b \otimes b'$ to $b.b'$. Hence the statement holds for $B \to B \otimes_A B$. But B is a faithfully flat A-module, so the statement holds. □

The constant sheaf on X_{et}

Let X be a quasi-compact scheme. We define the constant sheaf associated to a set S on X_{et} by $\mathcal{F}(U) = \prod_{\pi_0(U)} S$, where S is a set and $\pi_0(U)$ is the set of connected components of U.

The sheaf defined by a coherent $\mathcal{O}_X$-module

Let us recall that given a locally ringed space $(X,\mathcal{O}_X)$, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for every U open of X, with the restriction maps compatible with the module structures of $\mathcal{F}(U)$ and of $\mathcal{F}(V)$ for $V \subseteq U$. A morphism of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ is a morphism of sheaves such that for every open U of X, $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism. The kernel, cokernel, image and co-image of an $\mathcal{O}_X$-module homomorphism are again $\mathcal{O}_X$-modules. The tensor product sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of two $\mathcal{O}_X$-modules is defined to be the sheaf associated to the presheaf $U \to \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Now, let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then $f^{-1}\mathcal{G}$ is an $f^{-1}\mathcal{O}_Y$-module. We define the **inverse image of $\mathcal{G}$ by f** to be the tensor product:

$$f^*\mathcal{G} := f^{-1}G \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

For example, if X and Y are affine, defined by rings A and B respectively, then $\mathcal{G}$ is defined by a B-module M and $f^*\mathcal{G}$ corresponds to the A-module $M \otimes_B A$.

Now let A be a ring and M be an A-module. We define a sheaf M^c on Spec(A) as follows: Take an open set U of Spec(A), and set

$M^c(U) := \{s : U \to \coprod_{p \in U} M_p$ such that for all $p \in U, s(p) \in M_p$ satisfying the following:

there exist a neighborhood V of p in U and elements $m \in M$, $f \in A$ such that for all $q \in V$, $f \notin q$ and $s(q) = m/f$ $\}$.

Definition 3.2.6. Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called *quasi-coherent* if it has an open affine cover $U_i =$ Spec A_i such that for every i, there exists an A_i-module M_i with the restriction of $\mathcal{F}$ to U_i isomorphic (as a sheaf) to M_i^c. If M_i can be chosen to be of finite type for every i, then $\mathcal{F}$ is called *coherent*.

Now, let $\mathcal{M}$ be a coherent $\mathcal{O}_X$-module. We will construct a sheaf defined by $\mathcal{M}$ on X_{et} as follows: Let $\phi : U \to X$ be an étale morphism. Then $\phi^*\mathcal{M}$ is a coherent $\mathcal{O}_U$-module. Define

$$\mathcal{M}_{et}(U) := \Gamma(U, \phi^*\mathcal{M}). \tag{7}$$

It is a presheaf on X_{et} and to verify that it is indeed a sheaf, we proceed the same as Proposition 3.2.5 to prove that, for any faithfully flat morphism $B \to A$ and any B-module M, the sequence:

$$0 \to M \to M \otimes_B A \to M \otimes_B A \otimes_B A$$

is exact.

Skyscraper sheaves

Recall that a skyscraper sheaf on a topological space X is a sheaf $\mathcal{F}$ such that $\mathcal{F}_x = 0$ except for finitely many $x \in X$. Let X be a Hausdorff topological space, G an abelian group, let $x \in X$ we define a sheaf G^x by:

$$G^x(U) = \begin{cases} G & \text{if } x \in U; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that this sheaf depends on both x and G. Moreover the stalks of G^x at a point y of X are:

$$G^x_y = \begin{cases} G & \text{if } y = x; \\ 0 & \text{otherwise.} \end{cases}$$

We also have, for a sheaf of abelian groups $\mathcal{F}$ on X, from the definition of inductive limits that:

$$\mathrm{Hom}_{Sh}(\mathcal{F}, G^x) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}_x, G).$$

This above homomorphism comes from the fact that, to define a map from $\mathcal{F}_x$, which is the inductive limit of $\mathcal{F}(U)$ over all neighborhoods U of x in X, to G is the same as defining a family of morphisms from $\mathcal{F}(U)$ to G, which means, by definition, giving a natural transformation between $\mathcal{F}$ and G^x, and thus a morphism of sheaves $\mathcal{F} \to G^x$.

Now, let X be a variety over an algebraically closed field k, we will define a version of skyscraper sheaves on X_{et} as follows:

Let $\phi : U \to X$ be an étale morphism, let $x \in X$ and let G be an abelian group. Define:

$$G^x(U) := \bigoplus_{u \in \phi^{-1}(x)} G.$$

This is again a sheaf and its stalks vanish everywhere except at x, where it has stalk G. If $u \in U$ is in $\phi^{-1}(x)$ then (U,u) is an étale neighborhood of x. So given a sheaf $\mathcal{F}$ of abelian groups on X, we have a map $\mathcal{F}(U) \to \mathcal{F}_x$, and by composing with the stalks map $\mathcal{F}_x \to G$, we get a family of morphisms $\mathcal{F}(U) \to G$ for every étale neighborhood (U,u) of x, and hence a morphism of sheaves $\mathcal{F} \to G^x$. So again, we have:

$$\mathrm{Hom}_{Sh}(\mathcal{F}, G^x) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}_x, G).$$

Let X be a scheme and let $i : \bar{x} \to X$ be a geometric point of X such that $x := i(\bar{x})$ is closed. Let G be an abelian group. For any étale $\phi : U \to X$ is an étale morphism, we define:

$$G^{\bar{x}}(U) := \bigoplus_{\mathrm{Hom}_X(\bar{x},U)} G.$$

This is a sheaf on X_{et}. Let $\mathcal{F}$ be a sheaf on X_{et}, we have a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, G^{\bar{x}}) \to \mathrm{Hom}(\mathcal{F}_{\bar{x}}, G).$$

3.3 The category of sheaves on X_{et}

The presheaves of abelian groups on X_{et} are exactly the contravariant functors from the category of étale X-schemes to the category **Ab** of abelian groups. They form a category $\mathbf{Presh}(X_{et})$. The category of sheaves on X_{et}, $\mathbf{Sh}(X_{et})$ is the full subcategory of $\mathbf{Presh}(X_{et})$ whose objects are the sheaves of abelian groups on X_{et}. In section 2, we have stated that for every site X, the category $\mathbf{Sh}(X)$ is an abelian category has enough injectives, but we have not provided a proof. We will proceed here to show that $\mathbf{Sh}(X_{et})$ is an abelian category with enough injectives, for any étale site.

3.3.1 Exactness in $\mathbf{Sh}(X_{et})$

Definition 3.3.1. A morphism of sheaves $\alpha : \mathcal{F} \to \mathcal{F}'$ is called *locally surjective* if for every étale open set $U \to X$ and every $s' \in \mathcal{F}'(U)$, there exists an étale covering of U, $\{U_i \to U\}_{i\in I}$ and $s_i \in \mathcal{F}(U_i)$ such that $s' |_{U_i} = \alpha(U_i)(s_i)$ for every i.

Proposition 3.3.2. *Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism of sheaves. Then the following are equivalent:*

(a) α is locally surjective.

(b) α is an epimorphism in $\mathbf{Sh}(X_{et})$.

(c) For every geometric point $\bar{x} \to X$, the map on the stalks $\mathcal{F}_x \to \mathcal{F}'_x$ is surjective.

Proof. (a) $\Rightarrow$ (b)

Let $\beta : \mathcal{F}' \to \mathcal{S}$ be a morphism of sheaves such that $\beta \circ \alpha = 0$, we want to show that $\beta = 0$. Let $U \to X$ be an étale morphism and let $s' \in \mathcal{F}'(U)$. Since α is locally surjective then there exist $s_i \in \mathcal{F}(U_i)$ such that $\alpha(U_i)(s_i) = s'_{U_i}$. Now $\beta(s')_{U_i} = \beta(s' |_{U_i}) = \beta(\alpha s_i) = (\beta \circ \alpha)(s_i) = 0$ for every $i \in I$. By the sheaf property of $\mathcal{S}$, we have indeed that $\beta = 0$.

(b) $\Rightarrow$ (c)

Suppose that there exists a geometric point $\bar{x} \to X$ for which the map on the stalks $\alpha_{\bar{x}} : \mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}}$ is not surjective, and let $G := \mathrm{Coker}(\alpha_{\bar{x}})$. Then G is a non-zero abelian group. Moreover, we have (cf. discussion of skyscraper sheaves above), isomorphisms:

$$\mathrm{Hom}_{Sh}(\mathcal{F}, G^{\bar{x}}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}_{\bar{x}}, G)$$

and

$$\mathrm{Hom}_{Sh}(\mathcal{F}', G^{\bar{x}}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathcal{F}'_{\bar{x}}, G).$$

Hence, the composition:

$$\mathcal{F}_{\bar{x}} \xrightarrow{\alpha_{\bar{x}}} \mathcal{F}'_{\bar{x}} \longrightarrow G$$

(which is equal to zero since G is the cokernel of $\alpha_{\bar{x}}$) gives rise to the composition:

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{F}' \longrightarrow G^x.$$

which is zero by the above isomorphisms. But G^x is not always zero. This is a contradiction.

$(c) \Rightarrow (a)$

Let $U \to X$ be an étale morphism. And let $\bar{u} \to U$ be a geometric point of U. By composition with $U \to X$, we obtain a geometric point of X, that we denote by $\bar{x}$. Moreover, every étale neighborhood of $\bar{u}$ realizes an étale neighborhood of $\bar{x}$, via composition by $U \to X$. Hence, for every sheaf $\mathcal{F}$ on X_{et}, $\mathcal{F}_{\bar{x}} \simeq \mathcal{F}_{\bar{u}}$. Next, let $s \in \mathcal{F}'(U)$ and let $u \in U$. Let $i : \bar{u} \to U$ be a geometric point of U such that $i(\bar{u}) = u$. Now, (c) tells us that for every geometric point $\bar{u}$ of U, $\mathcal{F}_{\bar{u}} \to \mathcal{F}'_{\bar{u}}$ is surjective. Hence, there exists an étale map $V \to U$ whose image contains u such that the restriction of s to V is in the image of $\mathcal{F}(V) \to \mathcal{F}'(V)$. We take sufficiently many $u \in U$ and apply the same procedure to get a covering of U like Definition 3.3.1, as desired. □

Proposition 3.3.3. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

be a sequence of sheaves on X_{et}. The following are equivalent:

(a) The sequence is exact in $\mathbf{Sh}(X_{et})$.

(b) For every étale $U \to X$, the sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is exact.

(c) For every geometric point $\bar{x} \to X$, the sequence

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}}$$

is exact.

Proof. $(b) \Rightarrow (c)$ This implication comes from the fact that inductive limits are left exact in the category of abelian groups. $(c) \Rightarrow (b)$ Similar to $(c) \Rightarrow (a)$ in the proof of Proposition 3.3.2. $(a) \Leftrightarrow (b)$ This comes from the fact that the sheafification functor is exact, that we prove in the next section. □

Proposition 3.3.4. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a sequence of sheaves on X_{et}. The following are equivalent:

(a) The sequence is exact in $\mathbf{Sh}(X_{et})$.

(b) $\mathcal{F} \to \mathcal{F}''$ *is locally surjective, and for every étale* $U \to X$*, the sequence*

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$

is exact.

(c) *For every geometric point* $\bar{x} \to X$*, the sequence*

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$$

is exact.

Proof. Follows directly from the two above propositions. □

Theorem 3.3.5. *The category* $\mathbf{Sh}(X_{et})$ *is an abelian category.*

Proof. Since an isomorphism on the stalks is an isomorphism on the sheaf level, and since the category **Ab** of abelian groups is abelian, then the map from the co-image of a morphism of sheaves to its image is an isomorphism. □

3.3.2 The sheaf associated to a presheaf

The category $\mathbf{Sh}(X_{et})$ is the full subcategory of $\mathbf{Presh}(X_{et})$ whose objects are the sheaves on X_{et}. So, we have a fully faithful functor

$$i_{X_{et}} : \mathbf{Sh}(X_{et}) \to \mathbf{Presh}(X_{et}).$$

Recall that in Section 2, we stated that for every site X, the functor $i_X : \mathbf{Sh}(X) \to \mathbf{Presh}(X)$ admits a left adjoint which is exact. We describe here explicitly the left adjoint $i^{ad}_{X_{et}} : \mathbf{Presh}(X_{et}) \to \mathbf{Sh}(X_{et})$ of $i_{X_{et}}$, and we prove that it is exact.

Definition 3.3.6. Two sections s_1 and s_2 of a presheaf are said to be *locally equal* if $s_1 |_{U_i} = s_2 |_{U_i}$ for every U_i in some covering $\{U_i \to U\}$ of U.

Proposition 3.3.7. *Let* $\iota : \mathcal{F} \to \mathcal{F}^+$ *be a homomorphism from a presheaf* $\mathcal{F}$ *to a sheaf* $\mathcal{F}^+$ *on* X_{et}*, and suppose we have:*

(i) ι *is locally surjective, and*

(ii) *the only sections of* $\mathcal{F}$ *that have the same image in* $\mathcal{F}^+(U)$ *are the locally equal sections.*

Then

(a) $\mathcal{F}^+$*, endowed with the homomorphism* ι*, is the sheaf associated to the presheaf* $\mathcal{F}$*.*

(b) $\iota_{\bar{x}} : \mathcal{F}_{\bar{x}} \to \mathcal{F}^+_{\bar{x}}$ *is an isomorphism for every geometric point* $\bar{x} \to X$*.*

Proof. (a) Suppose we have a homomorphism i' from $\mathcal{F}$ to some sheaf $\mathcal{F}'$. We want to show that there exists a sheaf homomorphism $\alpha : \mathcal{F}^+ \to \mathcal{F}'$ such that $\iota \circ \alpha = i'$. We take $s \in \mathcal{F}^+(U)$. Since ι is locally surjective, there exists a covering $(U_i \to U)$ and $s_i \in U_i$ for every i, such that $\iota(s_i) = s\,|_{U_i}$, so the s_i's have the same image in $\mathcal{F}^+(U)$, hence they are locally equal by condition (ii). Therefore, $i'(s_i) \in \mathcal{F}'(U_i)$ is independent of the choice of s_i (since the s_i are locally equal), and moreover $i'(s_i)$ and $i'(s_j)$ agree on $\mathcal{F}'(U_i \times_U U_j)$. By the sheaf property of $\mathcal{F}'$, there exists a unique element t that restricts to $i'(s_i)$ for all i. We define: $\alpha : \mathcal{F}^+ \to \mathcal{F}' : s \mapsto t$. It is the desired homomorphism.

(b) Since ι is locally surjective then $\iota_{\bar{x}}$ is surjective on the stalks. For injectivity, take two elements s_1 and s_2 in $\mathcal{F}_{\bar{x}}$ mapping to the same element in $\mathcal{F}^+_{\bar{x}}$. Then s_1 and s_2 have the same image on some étale neighborhood U of $\bar{x}$. Therefore, they are locally equal. Taking limits gives the required result. ☐

Definition 3.3.8. Let $\mathcal{P}$ be a subpresheaf of a sheaf $\mathcal{F}$. We define the *subsheaf of* $\mathcal{F}$ *generated by* $\mathcal{P}$ to be:

$$\mathcal{P}'(U) := \{s \in \mathcal{F}(U)\text{: there exists a covering } \{U_i \to U\} \text{ such that } s\,|_{U_i} \in \mathcal{P}(U_i) \text{ for every } i\}.$$

Remark. The morphism of presheaves $\mathcal{P} \to \mathcal{P}'$ is locally surjective.

Now let $\mathcal{F}$ be a presheaf of abelian groups on X_{et}. For each $x \in X$, we choose a geometric point $\bar{x} \to X$ mapping to x and construct a skyscraper sheaf $\mathcal{F}^{\bar{x}}_{\bar{x}}$. Next, take $\mathcal{F}^* := \prod \mathcal{F}^{\bar{x}}_{\bar{x}}$ where the product is taken over all geometric points mapping to x, this $\mathcal{F}^*$ is a sheaf on X_{et}. We also have a natural map $i : \mathcal{F} \to \mathcal{F}^*$ satisfying condition (ii) of Proposition 3.3.7. Let $\mathcal{F}^+$ be the subsheaf of $\mathcal{F}^*$ generated by the image of $\mathcal{F}$ under i. Then $\mathcal{F}^+$ is the sheaf associated to the presheaf $\mathcal{F}$ (Proposition 3.3.7). We define the left adjoint functor of $i_{X_{et}} : \mathbf{Sh}(X_{et}) \to \mathbf{Presh}(X_{et})$ by:

$$i^{ad}_{X_{et}}(\mathcal{F}) = \mathcal{F}^+$$

for every presheaf $\mathcal{F}$ on X_{et}.

Theorem 3.3.9. *The functor* $i^{ad}_{X_{et}} : \mathbf{Presh}(X_{et}) \to \mathbf{Sh}(X_{et})$ *is exact.*

Proof. Since, by Proposition 3.3.7, $\mathcal{F}_{\bar{x}} \simeq \mathcal{F}^+_{\bar{x}}$, and since exactness on the stalks implies exactness on the sheaf level, then the statement holds. ☐

3.3.3 Direct and inverse images of sheaves on étale sites

In section 2 of this work, we have constructed the functor $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ and its left adjoint $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$, starting from a continuous functions of sites $f : X \to Y$. Here, we describe explicitly these functors for the étale sites case. We see first how a morphism of schemes induces a continuous function on the corresponding étale sites.

Let $f : Y \to X$ be a morphism of schemes. Then f induces a continuous function $f^{et} : Y_{et} \to X_{et}$ in the following way: Send an open $U \to X$ of X_{et} (an étale morphism of schemes) to the open $U \times_X Y \to Y$ in Y_{et}.

If $\mathcal{F}$ is a presheaf on Y_{et}, we then have

$$f_*^{et}\mathcal{F}(U) = \mathcal{F}(U \times_X Y).$$

By abuse of notation and for simplicity, we will denoted f_*^{et} by f_*.

We want to define a left adjoint for the functor f_*.

Define, for a presheaf $\mathcal{F}$ on X_{et}, and for $V \to Y$ étale,

$$\mathcal{F}'(V) := \varinjlim \mathcal{F}(U)$$

where the limit is over the commutative diagrams:

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with étale $U \to X$.

Definition 3.3.10. We define the inverse image sheaf $f^*\mathcal{F}$ to be the sheaf associated to the presheaf $\mathcal{F}'$ above.

Proposition 3.3.11. *f^* is the left adjoint to f_*.*

Proof. In [10], chapter I, section 8. □

Proposition 3.3.12. *Let $f : Y \to X$ be a morphism of schemes. The functor f^* defined above is exact (and hence f_* preserves injectives by Proposition 1.6.2).*

Proof. Let $i : \bar{x} \to X$ be a geometric point of X, and let $\mathcal{F}$ be a sheaf on X_{et}. We have, by comparing the limits in the definitions of i^* and that of $\mathcal{F}_{\bar{x}}$, that $(i^*\mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{x}}$. So, similarly, if $\bar{y} \to Y$ is a geometric point of Y, and $\bar{x}$ is the geometric point arising from it via $\bar{y} \to Y \to X$, we have,

$$(f^*\mathcal{F})_{\bar{y}} = i^*(f^*\mathcal{F})(\bar{y}) = \mathcal{F}_{\bar{x}}$$

Therefore f^* is an exact functor, since the above equality is true for any arbitrary geometric point $\bar{y}$ of Y. □

3.4 Étale cohomology

Theorem 3.4.1. *Let X be a scheme. The category $\mathbf{Sh}(X_{et})$ has enough injectives.*

Proof. For every x in X, we choose a geometric point $i_x : \bar{x} \to X$ such that $i_x(\bar{x}) = x$. Since the category **Ab** of abelian groups has enough injectives, then there exists an injective abelian group $I(x)$ together with an injection $\mathcal{F}_{\bar{x}} \hookrightarrow I(x)$. By Proposition 3.3.12, we have that $i_{x*}(I(x))$ is an injective sheaf. Call it I^x, and set $\mathcal{I} = \prod I^x$. $\mathcal{I}$ is again injective, being the product of injective sheaves. Finally, we get the following inclusion:

$$\mathcal{F} \hookrightarrow \prod \mathcal{F}_{\bar{x}}^{\bar{x}} \hookrightarrow \mathcal{I}$$

as desired. □

Brief description of the étale site of the spectrum of a field.

Let k be a field and let k^{sep} be a separable closure of k. Denote by G the Galois group $\mathrm{Gal}(k^{sep}/k)$. For a sheaf $\mathcal{F}$ on $(\mathrm{Spec}(k))_{et}$, we define

$$M_{\mathcal{F}} = \varinjlim \mathcal{F}(\mathrm{Spec}(k'))$$

where the direct limit is over all finite extensions of k in k^{sep}. Then, $M_{\mathcal{F}}$ is a discrete G-module. From the other side, for any discrete G-module M, we define

$$\mathcal{F}_M(A) = \mathrm{Hom}_G(H(A), k^{sep})$$

where $H(A)$ is defined to be the G-set $\mathrm{Hom}_{k-\mathrm{alg}}(A, k^{sep})$. Then $\mathcal{F}_M$ is a sheaf on $(\mathrm{Spec}(k))_{et}$. Moreover, the functors $\mathcal{F} \mapsto M_{\mathcal{F}}$ and $M \mapsto \mathcal{F}_M$ are equivalence of categories between the category of sheaves on $(\mathrm{Spec}(k))_{et}$ and the category of discrete G-modules. In particular, if k is separably closed, then the category of sheaves on $(\mathrm{Spec}(k))_{et}$ is equivalent to the category **Ab** of abelian groups. For a clearer description with proofs, see [11], Chapter II, Section 2.

Definition 3.4.2. Let X be a k-scheme, where k is a separably closed field. Consider the étale topology on X, and let $\mathcal{F}$ be a sheaf on X_{et}. We define the étale cohomology groups of X with values in the sheaf $\mathcal{F}$ by:

$$H^i(X_{et}, \mathcal{F}) := R^i c_* \mathcal{F}$$

$(i \geq 0)$, where $c : X \to \mathrm{Spec}(k)$.

This agrees with our definition in Section 2, since $\mathbf{Sh}((\mathrm{Spec}(k))_{et})$ is equivalent as a category to **Ab**.

4 Comparison of Cohomologies

In this section, we will present a comparison between cohomology of sheaves on different sites. We will illustrate this comparison through computations of cohomology for specific sites, by taking the Zariski topology and the étale topology on a scheme X and comparing the results of cohomology for the étale and Zariski sites. We begin by defining spectral sequences, our main tool for the intended comparison.

4.1 Spectral sequences

Definition 4.1.1. Let $a \geq 1$. An *a-th stage spectral sequence* in an abelian category $\mathcal{C}$ consists of the following:

- bigraded objects $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$ of $\mathcal{C}$, $r \geq a$.
- morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ in $\mathcal{C}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ satisfying:

$$E_{r+1}^{p,q} = ker(d_r^{p,q})/Im(d^{p-r,q+r-1}).$$

Let $E_r^{p,q}$ be a spectral sequence such that for every pair (p,q), the term $E_r^{p,q}$ stabilizes as $r \to \infty$. We consider a nonegative filtered complex in $\mathcal{C}$, $K^\bullet = F^0K^\cdot \supseteq F^1K^\bullet \supseteq \cdots$, that is a complex where $K^n = 0$ for $n < 0$, and where each of its objects admits a filtration, and consequently each of its cohomology objects admits a filtration. We write $\mathrm{Gr}^p(K^\bullet) := F^pK^\bullet/F^{p+1}K^\bullet$. Then we have

Proposition 4.1.2. *There exists a spectral sequence $E_r^{p,q}$ satisfying:*

- $E_0^{p,q} = \mathrm{Gr}^pK^{p+q}$*;*
- $E_1^{p,q} = \mathrm{H}^{p+q}(Gr^pK^\bullet)$*;*
- $E_r^{p,q} = \mathrm{Gr}^p\mathrm{H}^{p+q}(K^\bullet)$ *for sufficiently large r.*

Proof. See [4] ☐

We will call such a sequence a *first quadrant spectral sequence*, it satisfies $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$.

Let $E_r^{p,q}$ be a spectral sequence such that for every pair (p,q), the term $E_r^{p,q}$ stabilizes as $r \to \infty$ (for instance, if the morphisms $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ vanish for sufficiently large r), and denote this stable value by $E_\infty^{p,q}$, and let H^n be objects with finite filtrations $H^n = F^tH^n \supseteq \cdots \supseteq F^sH^n \supseteq 0$. We say that $E_r^{p,q}$ *converges* to H^n if

$$E_\infty^{p,q} = \mathrm{Gr}^pH^{p+q},$$

and we write

$$E_r^{p,q} \Rightarrow H^{p+q}.$$

Proposition 4.1.3. *Suppose we have a convergent spectral sequence*

$$E_r^{p,q} \Rightarrow H^{p+q}.$$

If $E_\infty^{p,q} = 0$ *except for* $p = 0$*, then* $H^n = E_\infty^{0,n}$*. Likewise, if* $E_\infty^{p,q} = 0$ *except for* $q = 0$*, then* $H^n = E_\infty^{n,0}$*.*

Proof. See [4]. □

Definition 4.1.4. A double complex $B^{\bullet,\bullet}$ consists of a bigraded object $B = \bigoplus_{p,q\in\mathbb{Z}} B^{p,q}$ together with morphisms $d : B^{p,q} \to B^{p+1,q}$ and $\delta : B^{p,q} \to B^{p,q+1}$ for every $p, q \in \mathbb{Z}$ such that $d^2 = \delta^2 = \delta \circ d + d \circ \delta = 0$.

We define for a double complex $B^{\bullet,\bullet}$ a complex $\mathrm{Tot}(B^{\bullet,\bullet})$, that we call its *total complex*, such that

$$\mathrm{Tot}^n B^{\bullet,\bullet} = \bigoplus_{p+q=n} B^{p,q},$$

and whose differential D is defined to be $D = \delta + d$.

The total complex $\mathrm{Tot}^n(B)$ admits two natural filtrations

$$'F^p\mathrm{Tot}^n B = \textstyle\bigoplus_{\substack{r+s=n \\ r\geq p}} B^{r,s},$$

$$''F^q\mathrm{Tot}^n B = \textstyle\bigoplus_{\substack{r+s=n \\ s\geq q}} B^{r,s}.$$

To these filtrations, we can assign, by Proposition 4.1.2, spectral sequences $'E_r^{p,q}$ and $''E_r^{p,q}$, respectively, satsifying

$$'E_0^{p,q} = B^{p,q},$$

$$''E_0^{p,q} = B^{q,p}.$$

For a double complex $B^{\bullet,\bullet}$ where $B^{p,q} = 0$ if $p < 0$ or $q < 0$ (called *first quadrant* double complex), the spectral sequences $'E_r^{p,q}$ and $''E_r^{p,q}$ both converge to $H^{p+q}(\mathrm{Tot}(B^{\bullet,\bullet}))$ (the $(p+q)$-th cohomology object of the complex $\mathrm{Tot}(B^{\bullet,\bullet})$).

4.2 The Grothendieck spectral sequence

The Grothendieck spectral sequence is used to describe the relationship between the right derived functors of the composite of functors and the composite of their right derived functors. Before we construct it, we need to define what it means to have a fully injective resolution of a complex.

Let $C^\bullet$ be a complex and let $0 \to C^\bullet \to I^{\bullet,\bullet}$ be an injective resolution of it, that is, a diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{0,1} & \xrightarrow{d^{0,1}} & I^{1,1} & \xrightarrow{d^{1,1}} & I^{2,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I^{0,0} & \xrightarrow{d^{0,0}} & I^{1,0} & \xrightarrow{d^{1,0}} & I^{2,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^{0} & \longrightarrow & C^{1} & \longrightarrow & C^{2} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

such that each column is an injective resolution for C^i, and each row is a complex. We define $Z^j(I^{\bullet,p}) = \ker(d^{j,p})$, $B^j(I^{\bullet,p}) = \operatorname{im}(d^{j-1,p})$, and $H^j(I^{\bullet,p}) = Z^j(I^{\bullet,p})/B^j(I^{\bullet,p})$. So we have complexes:

$$0 \to Z^p(C^\bullet) \to Z^p(I^{\bullet,0}) \to Z^p(I^{\bullet,1}) \to \cdots,$$
$$0 \to B^p(C^\bullet) \to B^p(I^{\bullet,0}) \to B^p(I^{\bullet,1}) \to \cdots,$$
$$0 \to H^p(C^\bullet) \to H^p(I^{\bullet,0}) \to H^p(I^{\bullet,1}) \to \cdots.$$

Definition 4.2.1. The complex $I^{\bullet,\bullet}$ is called a *fully injective resolution* of the complex $C^\bullet$ if the complexes defined above are injective resolutions for $Z^p(C^\bullet)$, $B^p(C^\bullet)$, and $H^p(C^\bullet)$, respectively.

Lemma 4.2.2. *Any complex in an abelian category with enough injectives admits a fully injective resolution.*

Proof. See [7], Chapter 20, Lemma 9.5. □

We construct now the Grothendieck spectral sequence. Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories with enough injectives and let $\mathcal{H}$ be an abelian category, suppose we have left exact functors

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{H}$$

then, we get

Theorem 4.2.3. *If for every injective object I of $\mathcal{C}$, $F(I)$ is G-acyclic, then for each object A of $\mathcal{C}$, there exists a spectral sequence*

$$E_2^{p,q} = (R^p G)(R^q F)(A) \Longrightarrow R^{p+q}(G \circ F)(A).$$

This sequence is called *the Grothendieck spectral sequence.*

Proof. Let A be an object of $\mathcal{C}$. Consider an injective resolution $0 \to A \to I^\bullet$ of A in $\mathcal{C}$ and apply F to it, so that we have a complex $F(I^\bullet)$ in $\mathcal{D}$. Consider a fully injective resolution $J^{\bullet,\bullet}$ of $F(I^\bullet)$ in $\mathcal{D}$ and apply G to it, so that we get a double complex $GJ^{\bullet,\bullet}$. We next consider the spectral sequences $'E_r^{p,q}$ and $''E_r^{p,q}$ associated to the double complex $GJ^{\bullet,\bullet}$, so we have $'E_0^{p,q} = GJ^{p,q}$, $'E_1^{p,q} = (R^q G)(F(I^p))$. But for every p, I^p is injective and by hypothesis, this gives us that $F(I^p)$ is G-acyclic, which means that $(R^q G)(F(I^p)) = 0$ except for $q = 0$ where it is equal to $(G \circ F)(I^p)$. Next, we look at $'E_2^{p,q}$. By definition, it is equal to 0 unless for $q = 0$ where it is equal to $H^p((G \circ F)(I^\bullet))$ which is equal to $R^p(G \circ F)(A)$. Hence, by Proposition 4.1.3, $'E_2^{p,q}$ converges to $H^{p+q}(\mathrm{Tot}(F(J^{\bullet,\bullet}))$, which is nothing but $R^{p+q}(G \circ F)(A)$. But since $\mathrm{Tot}(F(J^{\bullet,\bullet}))$ is a first quadrant bicomplex, then $''E_2^{p,q}$ also converges to $R^{p+q}(G \circ F)(A)$. Now, $''E_2^{p,q} = (R^p G)(R^q F)(A)$ (see [4]), and hence we have the desired result. □

4.3 The Leray spectral sequence

Let X and Y be two sites. If we have a continuous function of sites

$$X \xrightarrow{f} Y$$

then f induces a functor

$$\mathbf{Sh}(X) \xrightarrow{f_*} \mathbf{Sh}(Y)$$

that has a left adjoint

$$\mathbf{Sh}(Y) \xrightarrow{f^*} \mathbf{Sh}(X) \ .$$

Hence, as a corollary to Theorem 4.2.3, we have

Theorem 4.3.1. *For every $\mathcal{F} \in \mathbf{Sh}(X)$, there exists a spectral sequence*

$$E_2^{p,q} : H^p(Y, R^q f_* \mathcal{F}) \Longrightarrow H^{p+q}(X, \mathcal{F}) \tag{8}$$

Proof. We apply Theorem 4.2.3, with $F = f_*$ and $G = \Gamma_Y$. □

We call this sequence *the Leray spectral sequence.*

In particular, let X be a scheme, and consider the two sites X_{et} and X_{Zar} corresponding to the étale and Zariski topology on X, respectively. And let

$$\epsilon : X_{et} \to X_{Zar}$$

be the canonical continuous function between the Zariski and étale sites on X.

The corresponding Leray spectral sequence is

$$E_2^{p,q} : H^p(X_{Zar}, R^q\epsilon_*\mathcal{F}) \Longrightarrow H^{p+q}(X_{et}, \mathcal{F}). \tag{9}$$

We also have, for every sheaf $\mathcal{F}$ on X_{et} a morphism of cohomology functors

$$H^i(X_{Zar}, \epsilon_*\mathcal{F}) \to H^i(X_{et}, \mathcal{F})$$

for every $i \geq 0$ (cf. end of section 2.3).

Theorem 4.3.2. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module defined on X_{Zar}. By equation 7 (cf. section 3.2.1), it induces a sheaf $\mathcal{F}_{et}$ on X_{et}. Then, we have natural isomorphisms*

$$H^i(X_{Zar}, \mathcal{F}) \simeq H^i(X_{et}, \mathcal{F}_{et})$$

for every $i \geq 0$.

Proof. First, we have from equation 7, section 3.2.1, that for every étale $g : V \to X$:

$$\mathcal{F}_{et}(V) = (g^{-1}\mathcal{F} \otimes_{g^{-1}\mathcal{O}_X} \mathcal{O}_V)(V).$$

So, for every Zariski open U of X, if we denote by i the open immersion $U \to X$, we have

$$(\epsilon_*\mathcal{F}_{et})(U) = \mathcal{F}_{et}(U) = (i^{-1}\mathcal{F} \otimes_{i^{-1}\mathcal{O}_X} \mathcal{O}_U)(U) = (\mathcal{F} \mid_U)(U) = \mathcal{F}(U).$$

Hence, we have that $\epsilon_*\mathcal{F}_{et} = \mathcal{F}$. Next, we look at $R^q\epsilon_*\mathcal{F}_{et}$. Let $0 \to \mathcal{F}_{et} \to \mathcal{I}^\bullet$ be an injective resolution for $\mathcal{F}_{et}$. Denote the q-th cohomology of the complex $\epsilon_*\mathcal{I}^\bullet$ by $h^q(\epsilon_*\mathcal{I}^\bullet)$, then we have, for any $x \in X$,

$$(R^q\epsilon_*\mathcal{F})_x = (h^q(\epsilon_*\mathcal{I}^\bullet))_x = \varinjlim_{\substack{U \subseteq X \\ x \in U}} H^q(U_{et}, \mathcal{F}_{et}) = 0$$

since $H^q(U_{et}, \mathcal{F}_{et}) = 0$ (see [1], Tome 2, Corollary 4.4). Consequently, $R^q\epsilon_*\mathcal{F} = 0$ for $q > 0$ and hence, the second sheet of the spectral sequence

$$H^p(X_{Zar}, R^q\epsilon_*\mathcal{F}_{et}) \Longrightarrow H^{p+q}(X_{et}, \mathcal{F}_{et})$$

has nonvanishing terms only for $q = 0$, and hence we have the isomorphisms $H^i(X_{Zar}, \mathcal{F}) \simeq H^i(X_{et}, \mathcal{F}_{et})$. ☐

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