Exponential sums, hypersurfaces with many symmetries and Galois representations

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$$\psi_q : (\mathbb{F}_q, +) \longrightarrow (\mathbb{C}, \times), \qquad \psi_q(x) := \exp\left(\frac{2\pi i}{p}\,\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)\right).$$

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$$B_\ell(q; \alpha, \beta) := \sum_{x \in \mathbb{F}_q} \psi_q\big(\alpha x^{\ell+1} + \beta x\big) \in \mathbb{C}.$$

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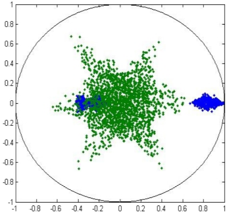
We want to understand how the normalized exponential sums

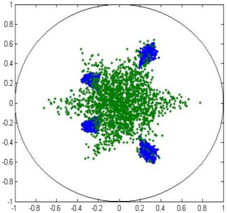
$$b_\ell(q; \alpha, \beta) := \frac{B_\ell(q; \alpha, \beta)}{\ell\sqrt{q}} \in \mathbb{D}$$

are distributed, on average, as $q \to \infty$.

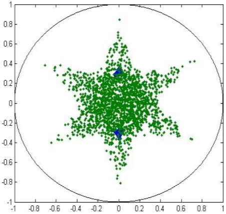
Example: Some of the multisets $\{b_3(p;\alpha,\beta) \mid \alpha,\beta \in \mathbb{F}_p, \alpha \neq 0\}$.



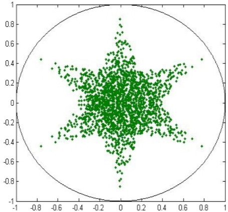
p = 1993



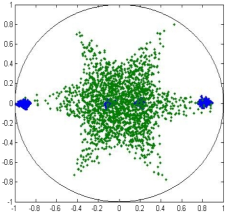
p = 1997



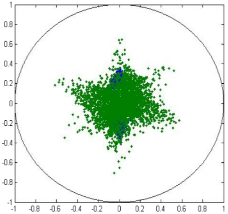
p = 1999



p = 2011



p = 2017



p = 2999

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Theorem (Livné) $$\lim_{q\to\infty} \mu_2(q) = \mu_{\mathrm{ST}},$$

where $\mu_{\mathrm{ST}} = \frac{2}{\pi}\sqrt{1-x^2}dx$ *is the Sato-Tate measure on* $[-1,1]$.

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Study the distributions $\mu_\ell(q)$ through their moments, which can be related to the number of $\mathbb{F}_q$-rational points on certain algebraic varieties.

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For general ℓ:

$$W_\ell^{m,n} : \quad \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_i = \sum_{i=1}^{m} x_i^{\ell+1} - \sum_{j=1}^{n} y_i^{\ell+1} = 0 \quad \text{in} \quad \mathbb{P}^{m+n-1}.$$

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Let $M_\ell^{m,n}(q)$ denote the (m,n)-th moment of $\mu_\ell(q)$, i.e.

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Lemma

$$M_\ell^{m,n}(q) = \frac{1}{\ell^N q^{N/2}} \left(|W_\ell^{m,n}(\mathbb{F}_q)| - \frac{q^{N-2}-1}{q-1} \right), \quad N = m+n.$$

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Lefschetz trace formula: $|W_\ell^{m,n}(\mathbb{F}_q)|$ can be studied via the action of Frobenius on the étale cohomology groups $H^\bullet(W_\ell^{m,n})$.

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In that case, the blow-up $\widetilde{W}_\ell^{m,n}$ along the singularities is smooth.

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Theorem

If $q = p^r$ with p inert in $\mathbb{Q}(\zeta_\ell)$,

$$M_\ell^{m,n}(q) = \pm \frac{1}{\ell^N} \dim H_{sub}^{N-4}(\widetilde{W}_\ell^{m,n}) + O\left(\frac{1}{\sqrt{q}}\right).$$

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Remark: The subprimitive cohomology vanishes unless

$$n \equiv m \bmod \ell \quad \text{and} \quad n \equiv m \bmod 2.$$

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Example: $H^4_{\mathrm{pr}}(W_7) \cong 2 \cdot \mathrm{sg} \oplus \theta_6 \oplus \theta_{14}$, where θ_6 and θ_{14} are irreducible representations of S_7 of degree 6 and 14, respectively.

Modularity and the Faltings-Serre method

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Numerical evidence: We seem to have

$$\mathrm{tr}(Frob_p) = a_p(f), \quad \text{for } p \neq 3, 5, 7,$$

$$f(q) = 1 + q^3 - 5q^5 + 7q^7 + q^9 - 13q^{11} + \cdots \in S_3(35, \varepsilon_{35}).$$

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Deligne's construction: $\rho \overset{?}{\sim} \rho_f$ (up to semi-simplification.)

Let K be a number field, S a finite set of primes of K, and

$$\rho_1, \rho_2 : G_K \longrightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$$

two continuous representations unramified outside S such that

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A subset $\Sigma \subseteq G_K$ is *sufficient* if

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Faltings-Serre method: There exists a *finite* sufficient set Σ depending only on S, ℓ and $\overline{\rho}_i$.

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Then Σ is sufficient.

Theorem

Suppose $\ell = 2$, and let H denote the common image of $\overline{\rho}_i$ in $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$. Let Σ be a subset of G_K which surjects onto $\overline{G}$, and in addition,

- *if $|H| \leq 2$: onto the greatest quotient of exponent 2 of G (Faltings-Serre-Livné)*
- *if $|H| = 3$: an element of order 6 in every intermediate quotient $\mathbb{Z}/2\mathbb{Z} \times H \cong \mathbb{Z}/6\mathbb{Z}$;*
- *if $H \cong S_3$: an element of order ≥ 3 in every intermediate quotient Q of the form $H \times \mathbb{Z}/2\mathbb{Z}$ or*

$$1 \longrightarrow V_4 \longrightarrow Q \longrightarrow \overline{G} \longrightarrow 1.$$

Then Σ is sufficient.